An infinite dimensional pursuit–evasion differential game with finite number of players

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Abstract

We study a pursuit–evasion differential game with finite number of pursuers and one evader in Hilbert space with geometric constraints on the control functions of players. We solve the game by presenting explicit strategies for pursuers which guarantee their pursuit as well as an strategy for the evader which guarantees its evasion.

Keywords: Differential game, Pursuit–evasion game, Geometric control constraints.

1 Introduction and preliminaries

Differential games and pursuit–evasion problems are investigated by many authors and significant researches are given by Isaacs [3] and Petrosyan [4].

Ibragimov and Salimi [1] study a differential game of optimal approach of countably many pursuers to one evader in an infinite dimensional Hilbert space with integral constraints on the controls of the players. Ibragimov et al. [2] study an evasion problem from many pursuers in a simple motion differential game with integral constraints. In [5] Salimi et al. investigate a differential game in which countably many dynamical objects pursue a single one. All the players perform simple motions. The duration of the game is fixed. The controls of a group of pursuers are subject to integral constraints and the controls of the other pursuers and the evader are subject to geometric constraints. The

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payoff of the game is the distance between the evader and the closest pursuer when the game is terminated. They construct optimal strategies for players and find the value of the game.

In the present paper, we solve a pursuit–evasion differential game with geometric constraints on the controls of players. In other words a pursuit of one player by finite number of dynamical players.

In the Hilbert space $\ell_2 = \{\alpha = (\alpha_k)_{k\in\mathbf{N}} \in \mathbf{R}^{\mathbf{N}} : \sum_{k=1}^{\infty} \alpha_k^2 < \infty\}$ with inner product $(\alpha, \beta) = \sum_{k=1}^{\infty} \alpha_k \beta_k$, the motions of the pursuers P_i and the evader E are defined by the equations:

$$\begin{aligned} &(P_i) : \dot{x}_i = u_i(t), \quad x_i(0) = x_{i0},, \quad i = 1, 2, \dots, m \\ &(E) : \dot{y} = v(t), \quad y(0) = y_0, \end{aligned} \tag{1.1}$$

where $x_i, x_{i0}, y, y_0 \in \ell_2$, $u_i = (u_{i1}, u_{i2}, \dots, u_{i\zeta}, \dots)$ is the control parameter of the pursuer P_i, and $v = (v_1, v_2, \dots, v_\zeta, \dots)$ is that of the evader E. In the following definitions $i = 1, 2, \dots, m$.

Definition 1. A function $u_i(\cdot)$, $u_i : [0, \infty) \to \ell_2$, such that $u_{i\zeta} : [0, \infty) \to R^1$, $\zeta = 1, 2, \dots$, are Borel measurable functions and

$$\|u_i(t)\| = \Big(\sum_{\zeta=1}^{\infty} u_{i\zeta}(t)^2 \, dt \Big)^{\frac{1}{2}} \le 1$$

is called an *admissible control of the ith pursuer.*

Definition 2. A function $v(\cdot)$, $v : [0, \infty) \to \ell_2$, such that $v_\zeta : [0, \infty) \to R^1$, $\zeta = 1, 2, \dots$, are Borel measurable functions and

$$\|v(t)\| = \Big(\sum_{\zeta=1}^{\infty} v_\zeta(t)^2 \, dt \Big)^{\frac{1}{2}} \le 1,$$

is called an *admissible control of the evader.*

Once the players' admissible controls $u_i(\cdot)$ and $v(\cdot)$ are chosen, the corresponding motions $x_i(\cdot)$ and $y(\cdot)$ of the players are defined as

$$x_i(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{i\zeta}(t), \dots), \quad y(t) = (y_1(t), y_2(t), \dots, y_\zeta(t), \dots),$$

$$x_{i\zeta}(t) = x_{i\zeta 0} + \int_0^t u_{i\zeta}(s) \, ds, \quad y_\zeta(t) = y_{\zeta 0} + \int_0^t v_\zeta(s) \, ds.$$

Definition 3. A function $U_i(t, x_i, y, v)$, $U_i : [0, \infty) \times \ell_2 \times \ell_2 \times \ell_2 \to \ell_2$, such that the system

$$\begin{aligned} &\dot{x}_i = U_i(t, x_i, y, v), \quad x_i(0) = x_{i0}, \\ &\dot{y} = v, \quad y(0) = y_0, \end{aligned}$$

has a unique solution $(x_i(\cdot), y(\cdot))$ for an arbitrary admissible control $v = v(t)$, $0 \le t < \infty$, of the evader E, is called a *strategy of the pursuer* P_i. A strategy U_i is said to be *admissible* if each control formed by this strategy is admissible.

Definition 4. A function $V(t, x_1, \dots, x_m, y)$, $V : [0,\infty) \times \underbrace{\ell_2 \times \ldots \times \ell_2}_{m+1} \to \ell_2$, such that the system of equations

$$\dot{x}_i = u_i, \quad x_i(0) = x_{i0},$$
$$\dot{y} = V(t, x_1, \dots, x_m, y), \quad y(0) = y_0,$$

has a unique solution $(x_1(\cdot), \dots, x_m(\cdot), y(\cdot))$ for arbitrary admissible controls $u_i = u_i(t)$, $0 \le t < \infty$, of the pursuers P_i, is called a *strategy* of the evader E. If each control formed by a strategy V is admissible, then the strategy V itself is said to be *admissible*.

2 Pursuit problem and its solution

Definition 5. If $x_i(\tau) = y(\tau)$ at some i and $\tau > 0$, then pursuit is considered complete.

Theorem 1. *Suppose the initial positions of the pursuers and the evader in the game (1.1) are different and for any non-zero vector* $p \in \ell_2$*, there is* $k \in \{1, 2, \dots, m\}$ *such that* $(y_0 - x_{k0}, p) < 0$*, then pursuit is complete.*

Proof. We define the pursuers' strategy as follow:

$$u_i(t) = v(t) - (v(t), e_i)\, e_i + e_i \left(1 - \|v(t)\|^2 + (v(t), e_i)^2\right)^{1/2}, \tag{2.1}$$

where $e_i = \dfrac{y_0 - x_{i0}}{\|y_0 - x_{i0}\|}$, $i = 1, 2, \dots, m$.
The above strategy is admissible. Indeed

$$\begin{aligned}\|u_i(t)\|^2 &= \|v(t) - (v(t),\ e_i)e_i\|^2 + 2\big(v(t) - (v(t),\ e_i)e_i,\ e_i\big(1 - \|v(t)\|^2 + (v(t),\ e_i)^2\big)^{1/2}\big) \\ &\quad + 1 - \|v(t)\|^2 + (v(t),\ e_i)^2 \\ &= \|v(t)\|^2 - 2(v(t),\ e_i)^2 + (v(t),\ e_i)^2 + (v(t),\ e_i)^2 + 1 - \|v(t)\|^2 \le 1.\end{aligned}$$

By (2.1), we have $y(t) - x_i(t) = e_i \Omega_i(t)$, where

$$\Omega_i(t) = \|y_0 - x_{i0}\| - \int_0^t \left(\left(1 - \|v(s)\|^2 + (v(s), e_i)^2\right)^{1/2} - (v(s), e_i)\right) ds.$$

We are going to show that $\Omega_i(\tau) = 0$, for some $i = 1, 2, \dots, m$, and $\tau > 0$.

It is clear that $\Omega_i(0) = \|y_0 - x_{i0}\| > 0$ for $i = 1, 2, \dots, m$.

Suppose that $\Omega(t) = \sum_{i=1}^m \Omega_i(t)$, thus

$$\Omega(t) = \sum_{i=1}^m \|y_0 - x_{i0}\| - \int_0^t \sum_{i=1}^m \left(\left(1 - \|v(s)\|^2 + (v(s), e_i)^2\right)^{1/2} - (v(s), e_i) \right) ds.$$

Obviously

$$\Lambda(v) = \sum_{i=1}^m \left(\left(1 - \|v\|^2 + (v, e_i)^2\right)^{1/2} - (v, e_i) \right) \geq 0.$$

Define

$$\Theta := \inf_{\|v\| \leq 1} \Lambda(v),$$

so $\Theta > 0$ or $\Theta = 0$.

We show that $\Theta \neq 0$. Assume by contradiction that $\Theta = 0$. Then there exists a minimizing sequence $\{v_n\}_n \subset \ell_2$ with $\|v_n\| \leq 1$ for the value $\Theta = 0$, i.e.,

$$\lim_{n\to\infty} \Lambda(v_n) = 0, \tag{2.2}$$

where

$$\Lambda(v) = \sum_{i=1}^m \left((1 - \|v\|^2 + (v, e_i)^2)^{\frac{1}{2}} - (v, e_i) \right) \geq 0.$$

On one hand, since the unit ball $B = \{v \in \ell_2 : \|v\| \leq 1\}$ is weakly compact (but not strongly compact due to the fact that ℓ_2 is an infinite dimensional Hilbert space), we may extract a subsequence (denoted in the same way) from $\{v_n\}$ which converges weakly to an element $v_0 \in B$, i.e.

$$v_n \overset{*}{\to} v_0 \text{ as } n \to \infty.$$

In particular, this fact implies that

$$\lim_{n\to\infty} (v_n, w) = (v_0, w), \ \forall w \in \ell_2. \tag{2.3}$$

On the other hand, since the real-valued sequence $\{\|v_n\|\}_n$ is bounded, up to some subsequence, we may assume that it converges to an element $c_0 \in [0, 1]$. We claim that $c_0 = 1$. Assume that $c_0 < 1$. Then, we have by (2.2) and (2.3) that

$$\begin{aligned}
0 &= \lim_{n\to\infty} \Lambda(v_n) \\
&= \lim_{n\to\infty} \sum_{i=1}^m \left((1 - \|v_n\|^2 + (v_n, e_i)^2)^{\frac{1}{2}} - (v_n, e_i) \right) \\
&= \sum_{i=1}^m \left((1 - c_0^2 + (v_0, e_i)^2)^{\frac{1}{2}} - (v_0, e_i) \right) \\
&> 0,
\end{aligned}$$

a contradiction. Therefore, $c_0 = 1$, i.e., $\lim_{n\to\infty} \|v_n\| = 1$.

Now, we come back again to (2.2), obtaining that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \Lambda(v_n) \\ &= \lim_{n\to\infty} \sum_{i=1}^{m} \Big((1 - \|v_n\|^2 + (v_n, e_i)^2)^{\frac{1}{2}} - (v_n, e_i) \Big) \\ &= \sum_{i=1}^{m} \Big(((v_0, e_i)^2)^{\frac{1}{2}} - (v_0, e_i) \Big) \\ &= \sum_{i=1}^{m} \left(|(v_0, e_i)| - (v_0, e_i) \right). \end{aligned}$$

Since every term under the sum is non-negative, we necessarily have that $|(v_0, e_i)| = (v_0, e_i)$ for all $i \in \{1, ..., m\}$. Therefore,

$$(v_0, e_i) = |(v_0, e_i)| \geq 0, \ \forall i \in \{1, ..., m\},$$

which is inconsistent with the hypothesis of the theorem, therefore $\Theta > 0$.

So,

$$\Omega(t) \leq \Omega(0) - \int_0^t \Theta \, ds = \Omega(0) - \Theta t,$$

therefore, in time $\eta = \frac{\Omega(0)}{\Theta}$ we have $\Omega(\eta) = \sum_{i=1}^{m} \Omega_i(\eta) \leq 0$, and then $\Omega_i(\tau) = 0$, for some $i = 1, 2, \ldots, m$, $\tau \in (0, \eta]$ and pursuit is complete. $\square$

3 Evasion problem and its solution

Definition 6. If there exists a strategy of the evader such that $x_i(t) \neq y(t)$, $t > 0$, then evasion is possible.

Theorem 2. *Suppose the initial positions of the pursuers and the evader in the game (1.1) are different and there exists a non-zero vector $p \in \ell_2$ such that $\|p\| = 1$ and $(y_0 - x_{i0}, p) \geq 0$, $i \in \{1, 2, \ldots, m\}$, then evasion is possible.*

Proof. We define the evader's strategy as follow:

$$v(t) = p, \quad t \geq 0. \tag{3.1}$$

Obviously the above strategy is admissible.

We have

$$(y(t) - x_i(t), p) = (y_0 - x_{i0}, p) + \int_0^t (v(s), p) \ ds - \int_0^t (u_i(s), p) \ ds.$$

By taking strategy (3.1) we obtain

$$(y(t) - x_i(t), p) = (y_0 - x_{i0}, p) + \int_0^t [1 - (u_i(s), p)] \, ds.$$

Let's assume that evasion is not possible, so there are $\tau > 0$ and $k \in \{1, 2, \dots, m\}$ such that $y(\tau) = x_k(\tau)$. Then

$$(y(\tau) - x_k(\tau), p) = (y_0 - x_{k0}, p) + \int_0^\tau [1 - (u_k(s), p)] \, ds = 0.$$

By the assumption of the theorem, $(y_0 - x_{k0}, p) \geq 0$. On the other hand $\|u_i(t)\| \leq 1$, and then

$$|(u_i(t), p)| \leq \|u_i(t)\| \cdot \|p\| \leq 1, \quad t \in [0, \tau],$$

so $(u_i(t), p) \leq 1$. Thus $(y_0 - x_{k0}, p) = 0$ and then

$$\int_0^\tau [1 - (u_k(s), p)] \, ds = 0.$$

From the above equality we obtain $1 - (u_k(s), p) = 0$, $s \in [0, \tau]$, almost everywhere. Hence $(u_k(s), p) = 1$, and $u_k(s) = p$, $s \in [0, \tau]$.

Therefore

$$y(\tau) - x_k(\tau) = y_0 + \int_0^\tau p \, ds - x_{k0} - \int_0^\tau p \, ds = y_0 - x_{k0} = 0,$$

which is a contradiction with the initial positions of the pursuers and the evader. So $x_i(t) \neq y(t)$, $i \in \{1, 2, \dots, m\}$, $t > 0$. In other words, evasion of the evader from all the pursuers is possible. $\square$

4 Conclusion

We considered a pursuit–evasion problem with finite number of pursuers and one evader in the Hilbert space ℓ_2. The controls of pursuers and the evader are subject to geometric constraints. We constructed admissible strategies for pursuers which guarantee capture of the evader as well as an admissible strategy for the evader which guarantees evasion from all pursuers.

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